

Bezier Curves

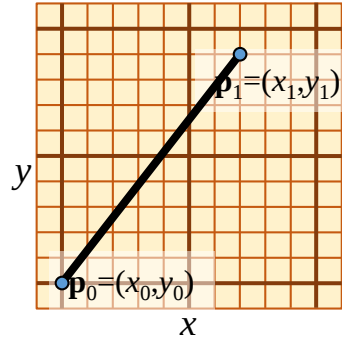
CS 418

Interactive Computer Graphics

John C. Hart

Linear Interpolation

$$\mathbf{p}(t) = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

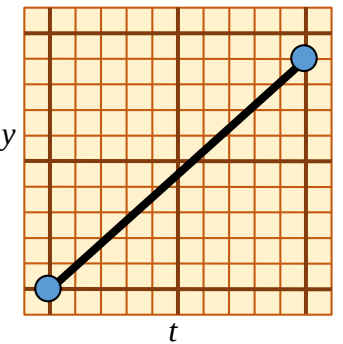
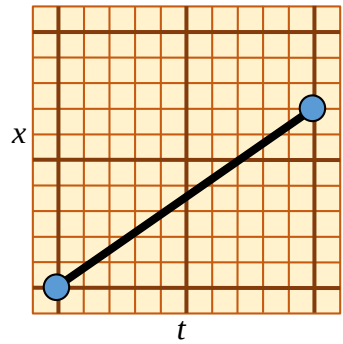
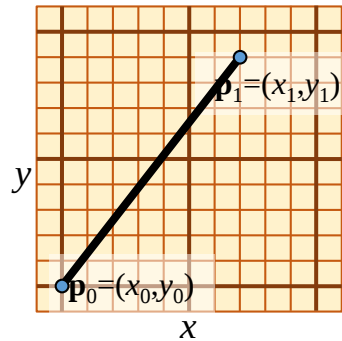


Linear Interpolation

$$\mathbf{p}(t) = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$x(t) = (1-t) x_0 + t x_1$$

$$y(t) = (1-t) y_0 + t y_1$$

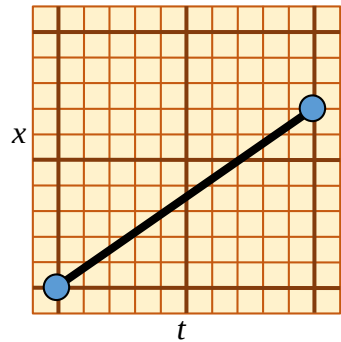
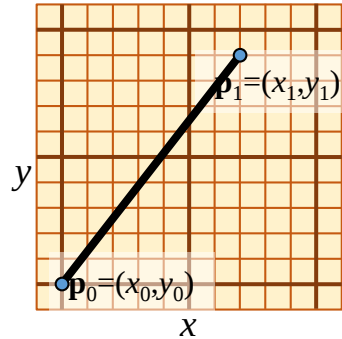


Linear Interpolation

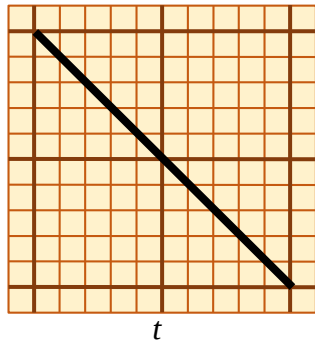
$$\mathbf{p}(t) = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$x(t) = (1-t) x_0 + t x_1$$

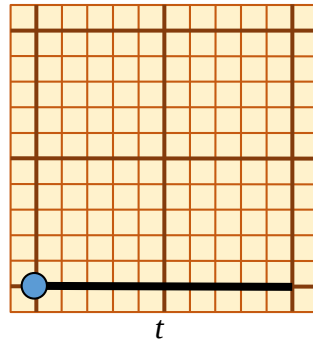
$$y(t) = (1-t) y_0 + t y_1$$



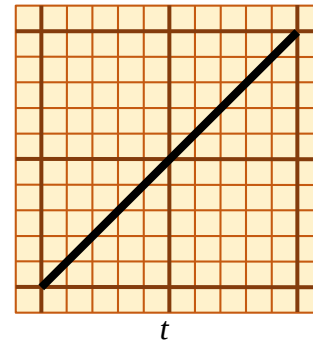
$1-t$



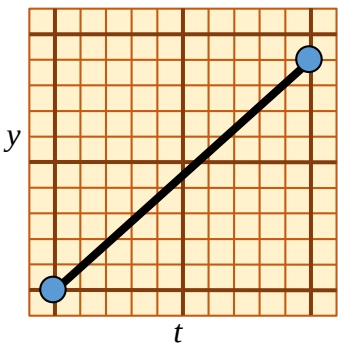
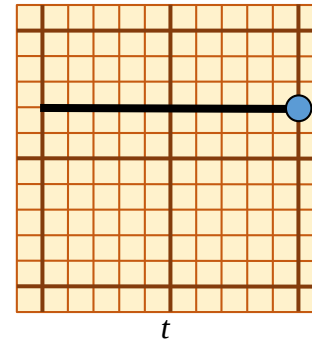
$* x_0$



$+ t$



$* x_1$

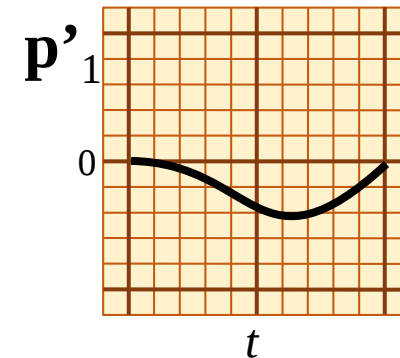
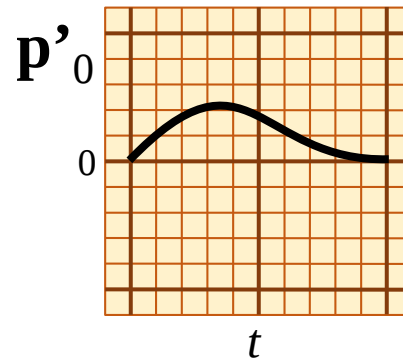
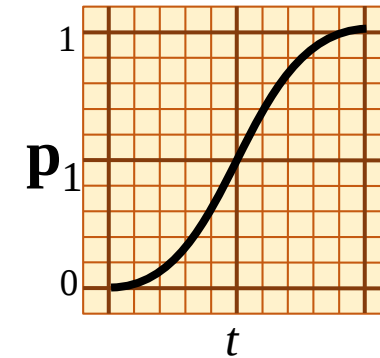
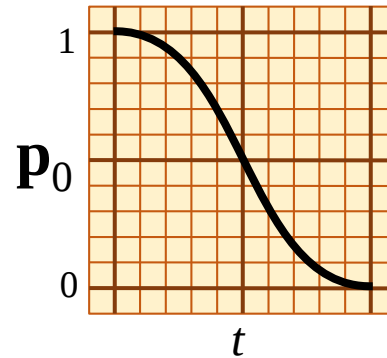
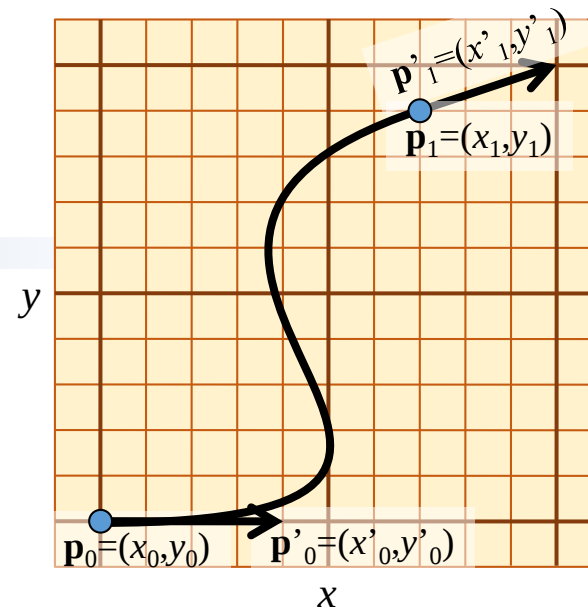


Hermite Interpolation

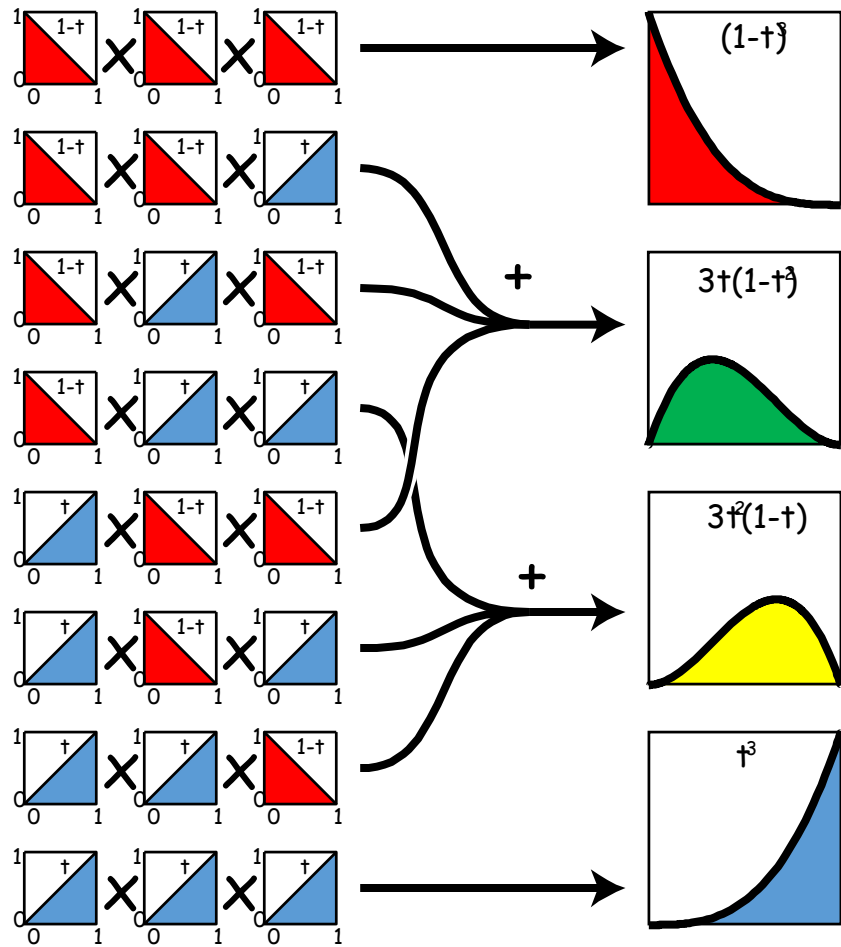
$$\mathbf{p}(t) = \begin{pmatrix} (2\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}'_0 + \mathbf{p}'_1) \\ (-3\mathbf{p}_0 + 3\mathbf{p}_1 - 2\mathbf{p}'_0 - \mathbf{p}'_1) \\ \mathbf{p}'_0 \\ \mathbf{p}_0 \end{pmatrix} \begin{pmatrix} t^3 + \\ t^2 + \\ t + \\ (1) \end{pmatrix}$$

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}'_0 \\ \mathbf{p}'_1 \end{bmatrix}$$

$$\mathbf{p}(t) = \begin{pmatrix} (2t^3 - 3t^2 + 1) \mathbf{p}_0 + \\ (-2t^3 + 3t^2) \mathbf{p}_1 + \\ (t^3 - 2t^2 + t) \mathbf{p}'_0 + \\ (t^3 - t^2) \mathbf{p}'_1 \end{pmatrix}$$



Bernstein Polynomials



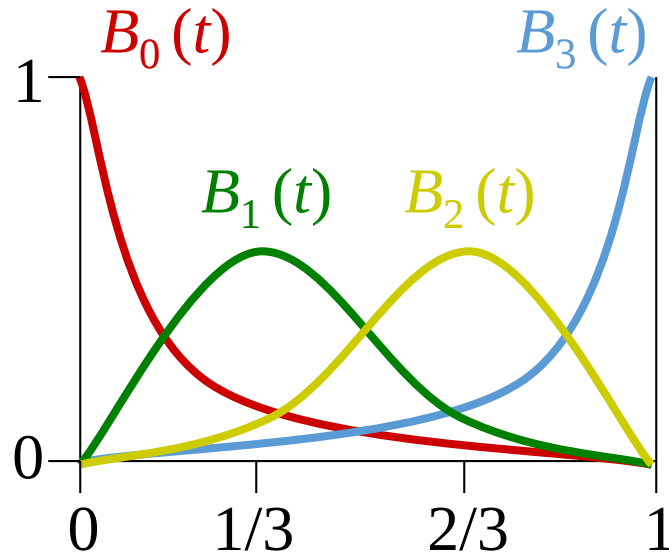
Bernstein Interpolation

$$B_0^3(t) = (1-t)^3$$

$$B_1^3(t) = 3(1-t)^2 t$$

$$B_2^3(t) = 3(1-t) t^2$$

$$B_3^3(t) = t^3$$



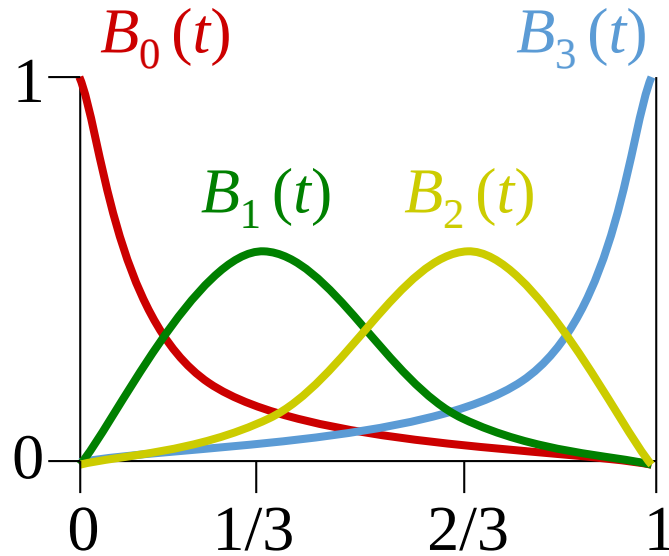
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$$B_0^3(t) = (1-t)^3$$

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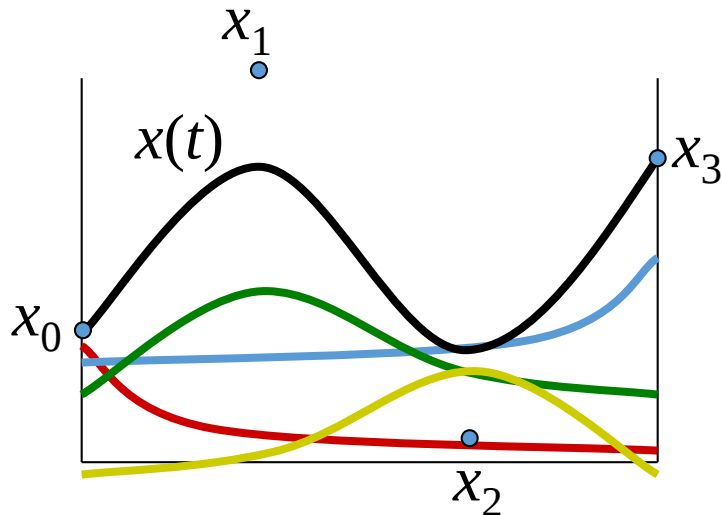


$$x(t) = B_0(t) x_0 +$$

$$B_1(t) x_1 +$$

$$B_2(t) x_2 +$$

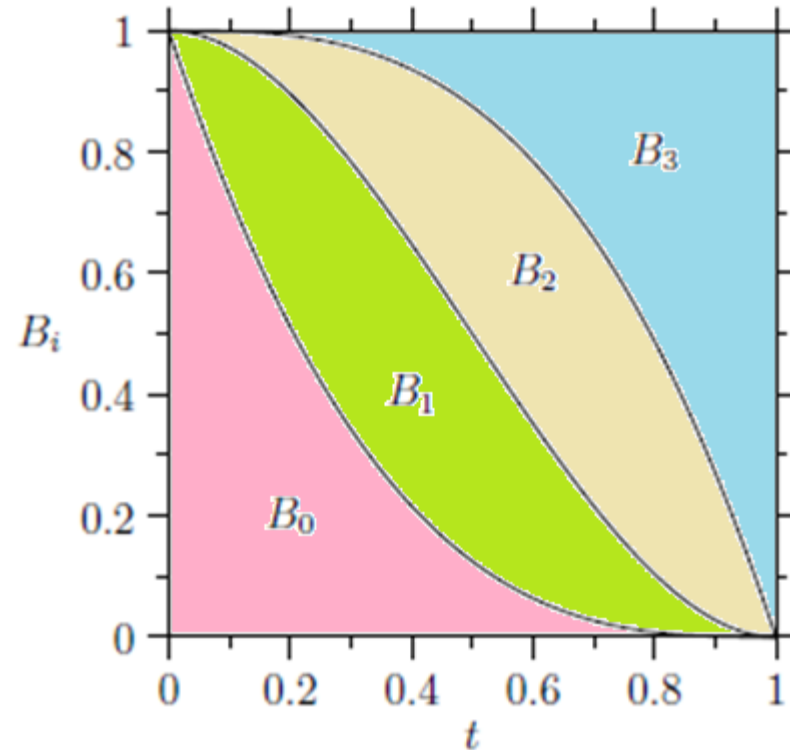
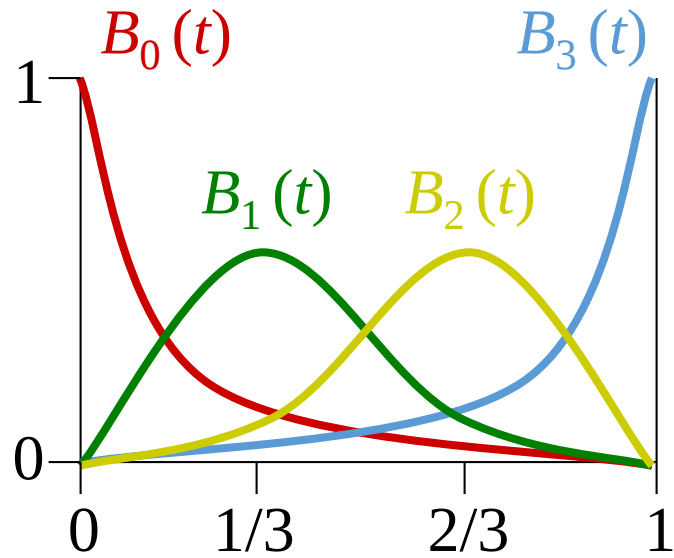
$$B_3(t) x_3$$



Bernstein Polynomials

- Partition of unity
 - Sum to one for any t in $[0,1]$

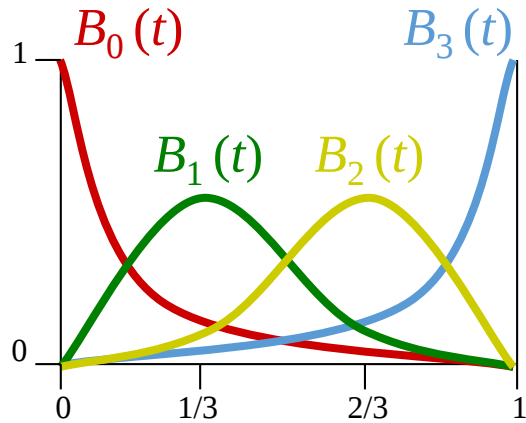
$$\sum_{i=0..n} B_i^n(t) = 1$$



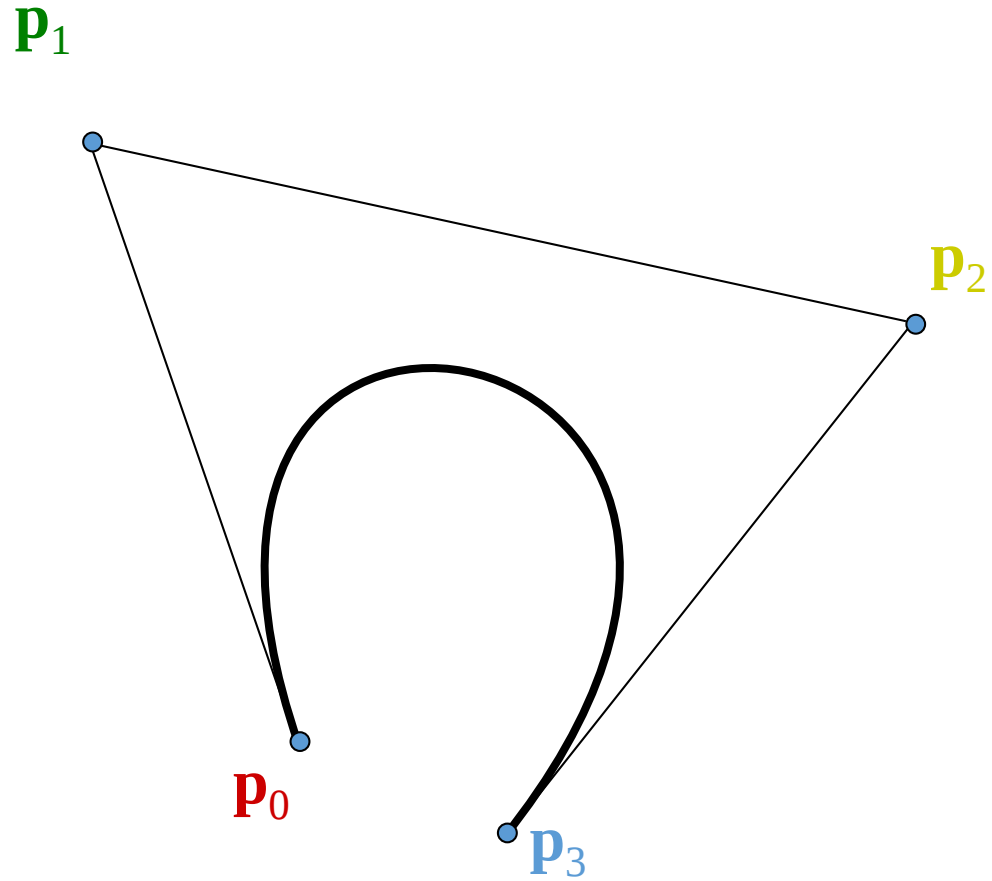
Cubic Bezier Curves

- Bernstein basis applied to points

$$\mathbf{p}(t) = \sum_i \binom{3}{i} t^i (1-t)^{3-i} \mathbf{p}_i$$



- Bezier curve specified by four *control* points including two *endpoints*



Cubic Bezier Curves

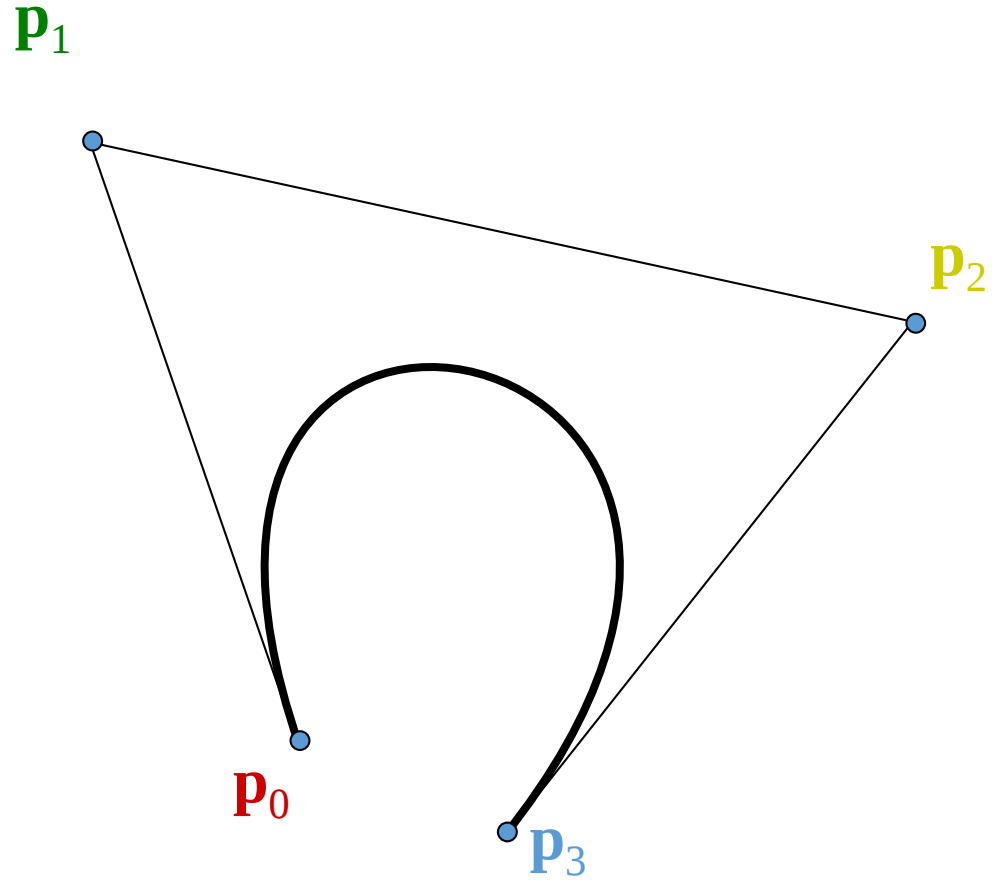
- Bernstein basis applied to points

$$\mathbf{p}(t) = \sum_i \binom{3}{i} t^i (1-t)^{3-i} \mathbf{p}_i$$

- Affine invariance:

Let M be a 4x4 transformation

Then $M \mathbf{p}(t) = \sum_i B_i(t) M \mathbf{p}_i$

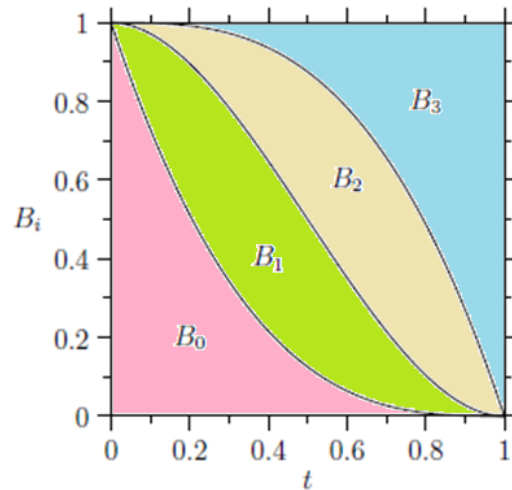
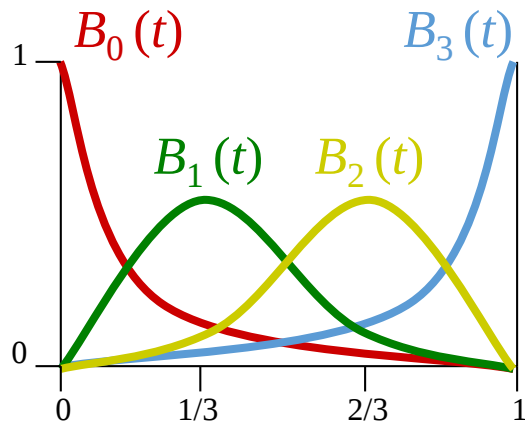


Cubic Bezier Curves

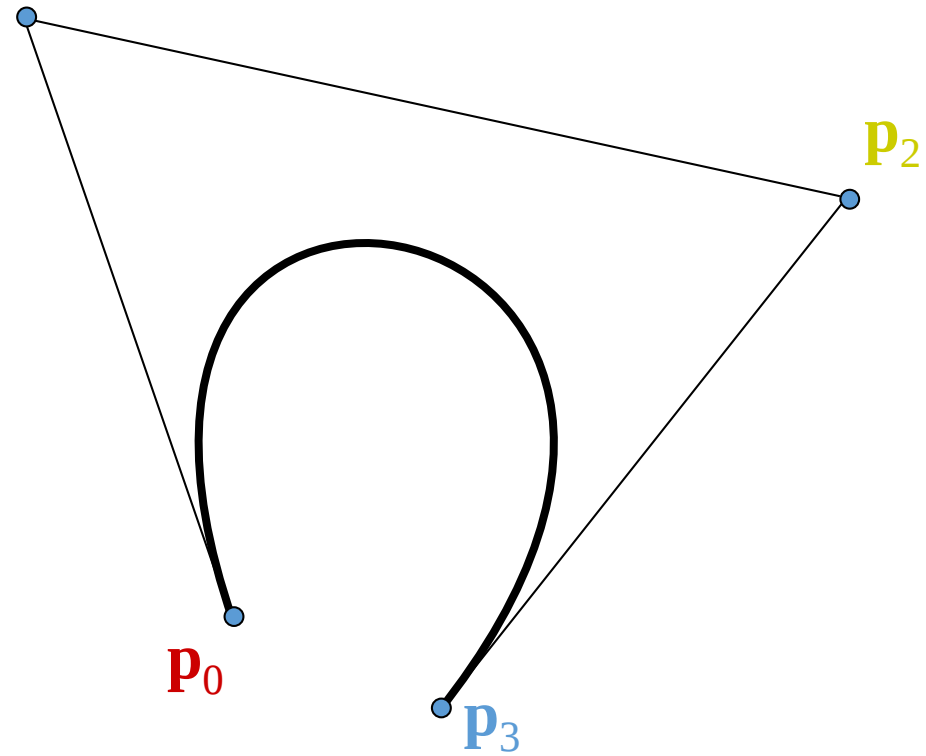
- Bernstein basis applied to points

$$\mathbf{p}(t) = \sum_i \binom{3}{i} t^i (1-t)^{3-i} \mathbf{p}_i$$

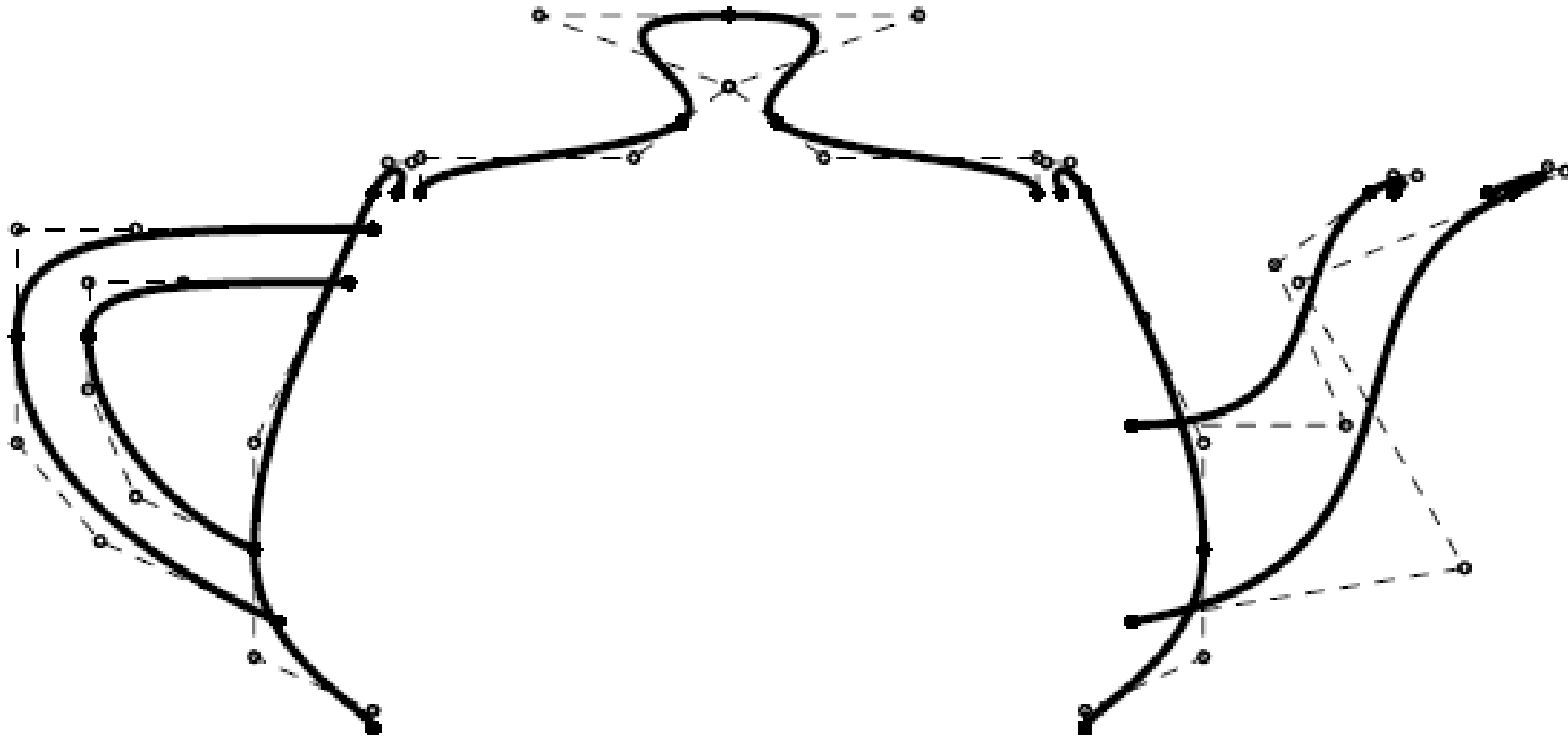
- Curve entirely contained in the convex hull of the control points



$\mathbf{p}_1$



Modeling with Bezier Curves



Bezier $\equiv$ Hermite

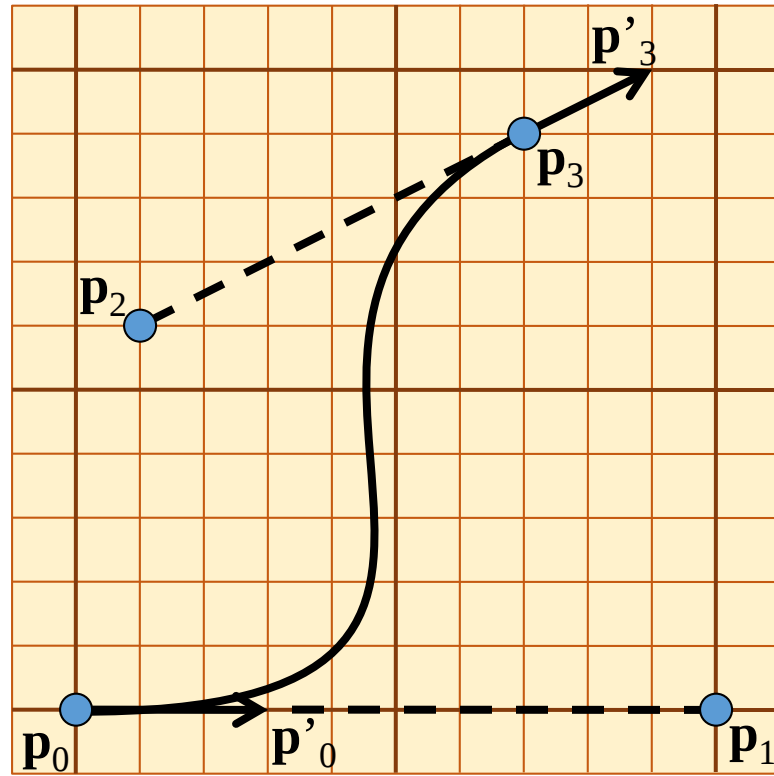
$$\mathbf{p}_1 = \mathbf{p}_0 + 3 \mathbf{p}'_0, \mathbf{p}_2 = \mathbf{p}_3 - 3 \mathbf{p}'_3$$

- Bezier

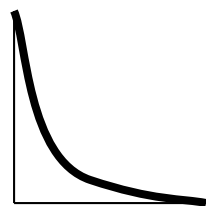
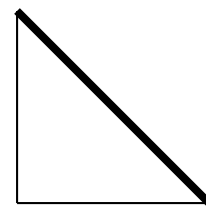
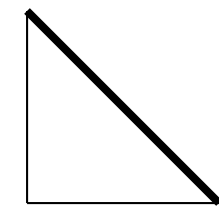
$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -3 & 1 \\ 2 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

- Hermite

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_3 \\ \mathbf{p}'_0 \\ \mathbf{p}'_3 \end{bmatrix}$$

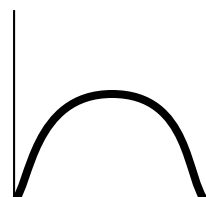
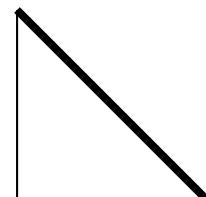
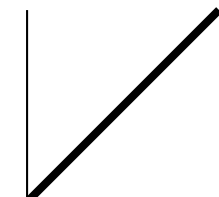
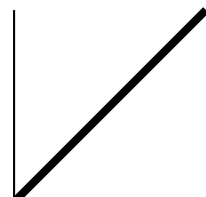
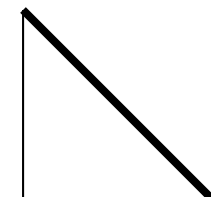


Building Bernsteins

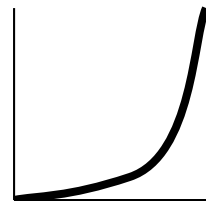
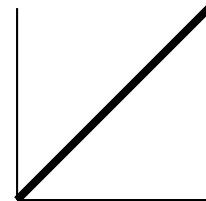
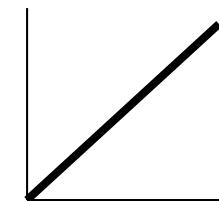

 $=$



$$B_0^2(t) = (1-t) B_0^1(t)$$

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$


 $=$


 $+$



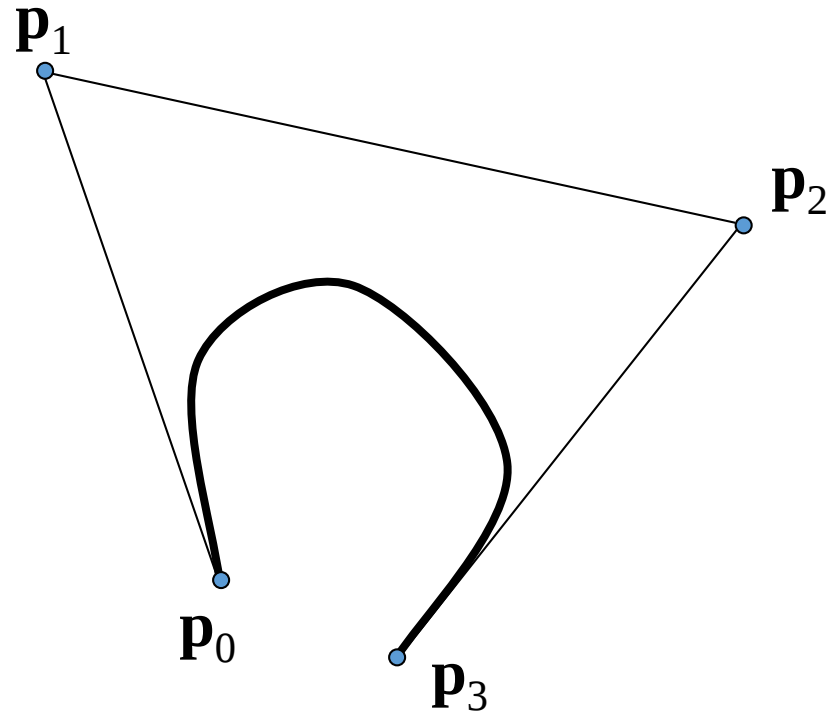
$$B_1^2(t) = (1-t) B_1^1(t) + t B_0^1(t)$$


 $=$



$$B_2^2(t) = t B_1^1(t)$$

de Casteljau Algorithm

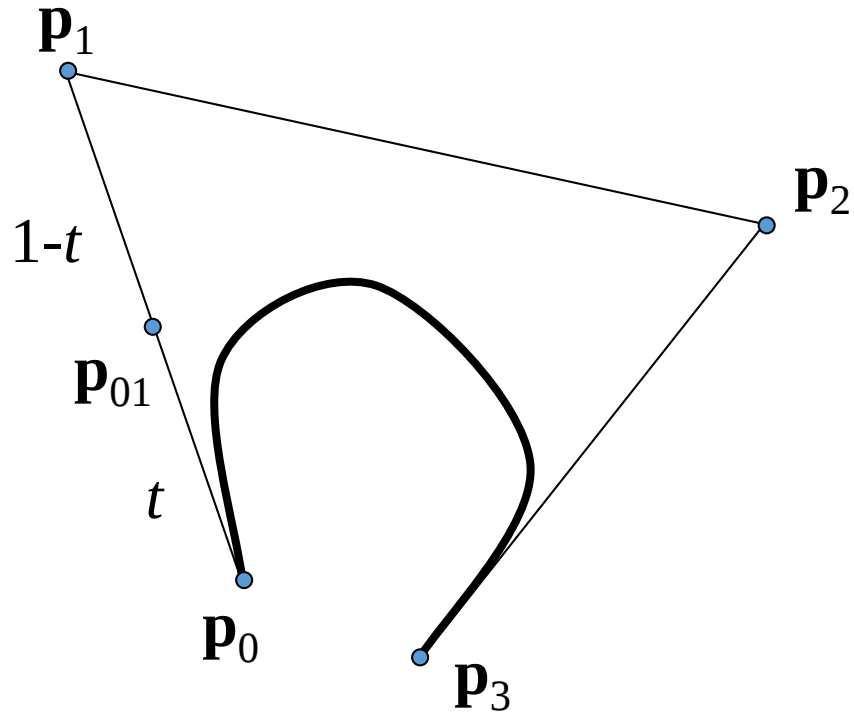
- Cascading lerps



de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

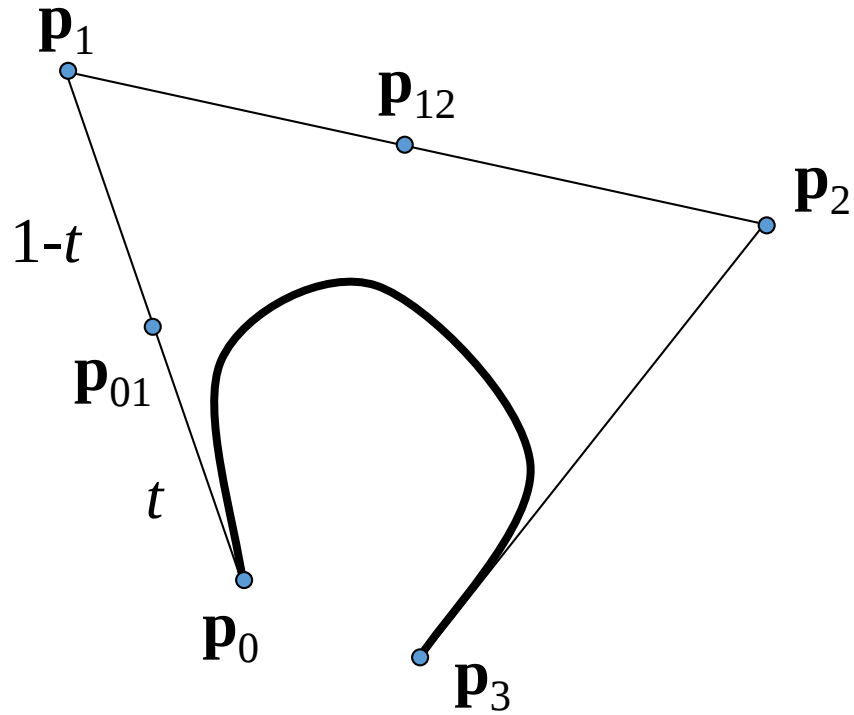


de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$



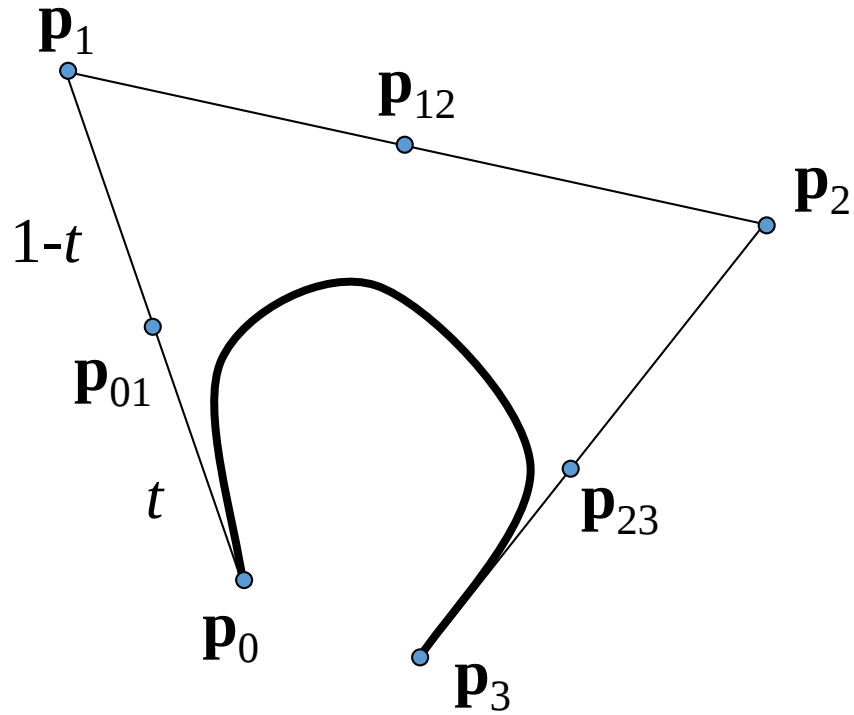
de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t) \mathbf{p}_2 + t \mathbf{p}_3$$



de Casteljau Algorithm

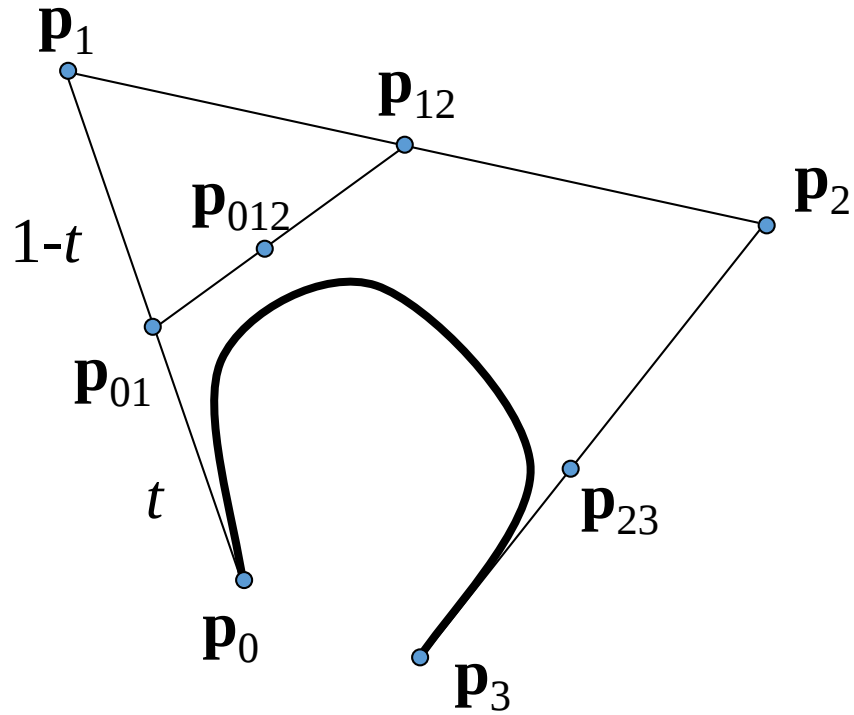
- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t) \mathbf{p}_2 + t \mathbf{p}_3$$

$$\mathbf{p}_{012} = (1-t) \mathbf{p}_{01} + t \mathbf{p}_{12}$$



de Casteljau Algorithm

- Cascading lerps

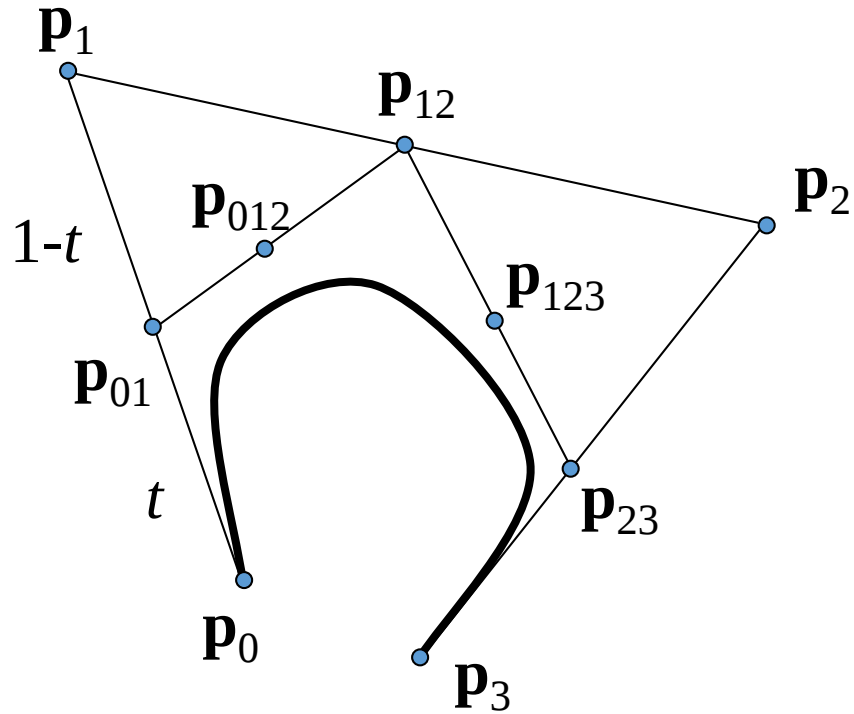
$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t) \mathbf{p}_2 + t \mathbf{p}_3$$

$$\mathbf{p}_{012} = (1-t) \mathbf{p}_{01} + t \mathbf{p}_{12}$$

$$\mathbf{p}_{123} = (1-t) \mathbf{p}_{12} + t \mathbf{p}_{23}$$



de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

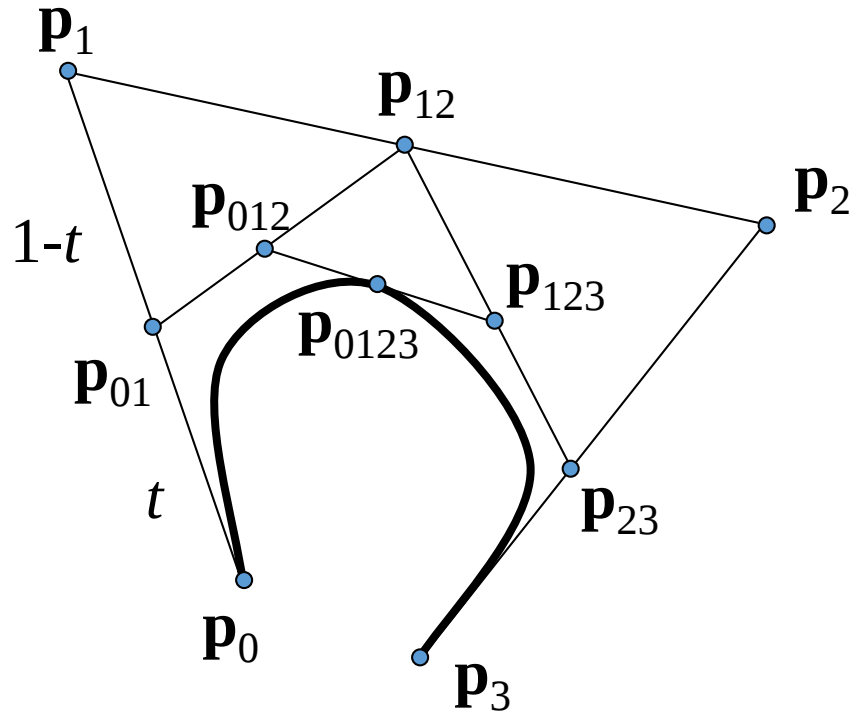
$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t) \mathbf{p}_2 + t \mathbf{p}_3$$

$$\mathbf{p}_{012} = (1-t) \mathbf{p}_{01} + t \mathbf{p}_{12}$$

$$\mathbf{p}_{123} = (1-t) \mathbf{p}_{12} + t \mathbf{p}_{23}$$

$$\mathbf{p}_{0123} = (1-t) \mathbf{p}_{012} + t \mathbf{p}_{123}$$



de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t) \mathbf{p}_2 + t \mathbf{p}_3$$

$$\mathbf{p}_{012} = (1-t) \mathbf{p}_{01} + t \mathbf{p}_{12}$$

$$\mathbf{p}_{123} = (1-t) \mathbf{p}_{12} + t \mathbf{p}_{23}$$

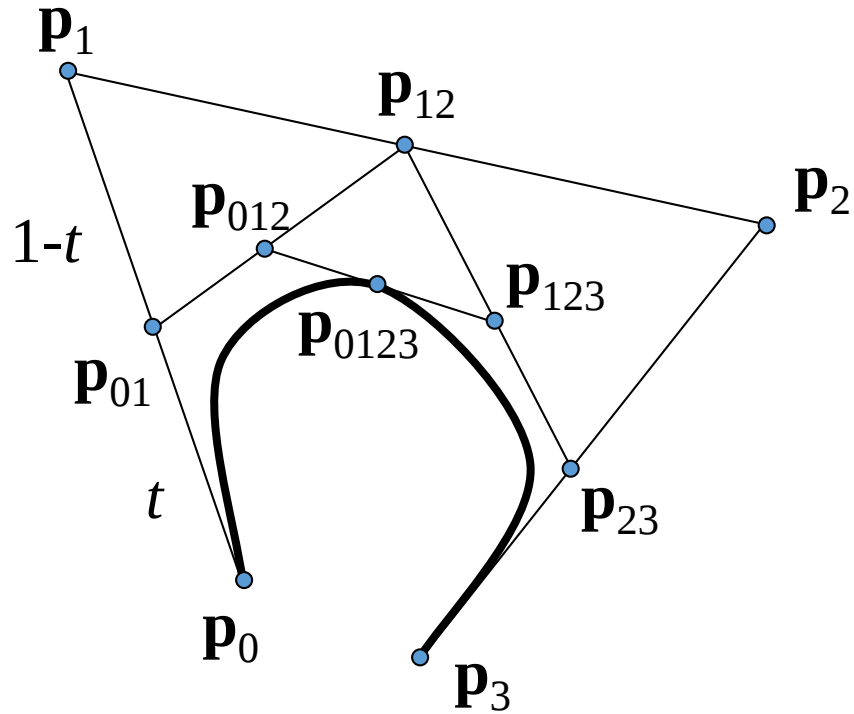
$$\mathbf{p}_{0123} = (1-t) \mathbf{p}_{012} + t \mathbf{p}_{123}$$

- Subdivides curve at $\mathbf{p}_{0123}$

- $\mathbf{p}_0 \mathbf{p}_{01} \mathbf{p}_{012} \mathbf{p}_{0123}$

- $\mathbf{p}_{0123} \mathbf{p}_{123} \mathbf{p}_{23} \mathbf{p}_3$

- Repeated subdivision converges to curve



de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1-t) \mathbf{p}_0 + t \mathbf{p}_1$$

$$\mathbf{p}_{12} = (1-t) \mathbf{p}_1 + t \mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t) \mathbf{p}_2 + t \mathbf{p}_3$$

$$\mathbf{p}_{012} = (1-t) \mathbf{p}_{01} + t \mathbf{p}_{12}$$

$$\mathbf{p}_{123} = (1-t) \mathbf{p}_{12} + t \mathbf{p}_{23}$$

$$\mathbf{p}_{0123} = (1-t) \mathbf{p}_{012} + t \mathbf{p}_{123}$$

- Subdivides curve at $\mathbf{p}_{0123}$

- $\mathbf{p}_0 \mathbf{p}_{01} \mathbf{p}_{012} \mathbf{p}_{0123}$

- $\mathbf{p}_{0123} \mathbf{p}_{123} \mathbf{p}_{23} \mathbf{p}_3$

- Repeated subdivision converges to curve

